

## New representation of two-loop propagator and vertex functions

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G.P.O. Box 252C, Hobart, 7001, Australia***Abstract**

We present a new method of calculating scalar propagator and vertex functions in the two-loop approximation, for arbitrary masses of particles. It is based on a double integral representation, suitable for numerical evaluation. Real and imaginary parts of the diagrams are calculated separately, so that there is no need to use complex arithmetics in the numerical program.

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# 1 Introduction

When the existence of the top quark becomes officially confirmed by Tevatron experiments, a chapter of particle physics history will be concluded. It has been the opinion of the vast majority of the physics community that the top quark probably exists, as required by the family structure of the standard model.

There is, however, no such unanimity as to what comes next. Supersymmetry, technicolor, compositeness, are just a few of theoretical ideas put forward to extend the standard model, and it will take a new generation of experiments to give us new insights.

On the other hand, when mass of the top quark becomes known, we will have enough information to re-interpret the existing results of high precision experiments in terms of bounds on the new physics. It will then be possible to discard some models and concentrate on the remaining ones in the future theoretical and experimental work. For this purpose it is crucial to know the theoretical predictions of the standard model with sufficient accuracy, and in many cases this requires going to the two-loop order of perturbation theory.

In the recent past considerable progress has been achieved in computing massive two-loop Feynman diagrams [1, 2, 3, 4, 5, 6, 7, 8]. Nevertheless, none of the existent approaches is able to give numerical results to the three-point functions in arbitrary kinematical regions, as needed, for example, in the determination of two-loop corrections to partial rates of  $Z$  boson decays.

The purpose of this paper is to present a solution to the problem of calculating 2- and 3-point functions in a general mass case. In the present work we restrict our investigations to scalar functions, which enables us to discuss in detail the analytical properties of a given diagram. The treatment of tensor functions will be presented in a forthcoming paper.

Let us briefly summarize the results achieved so far. In the most flexible approach one can represent the graph using Feynman parameters and let a computer do the work. With sufficient memory and CPU time available one can obtain stable numerical results. Such an approach is advocated by [1]. The advantage of this method is that one can treat rather general mass configurations of Feynman diagrams. On the other hand, the accuracy which one can obtain may not be sufficient in realistic calculations. Another problem is the extensive use of computer power. The four- or five-fold integral representations used by this method are on the edge of what is available for Monte Carlo integrations nowadays.

Another approach is to use unitarity relations to find the imaginary part of the two-loop diagram and then employ dispersion relations to compute the real part. The first step is rather easy, since excellent tools have been developed to calculate one-loop subdiagrams (see e.g. [9, 10]). However, the real part is difficult to calculate numerically, and additional complications may arise in the presence of anomalous thresholds.

More sophisticated approaches use the idea of expanding the integrand in some appropriate manner. The hope is that the resulting terms are integrable and the generated series converges.

In this line, there are two main approaches which have been described in the literature. Both use asymptotic expansions in some appropriate kinematical variable, e.g.  $q^2$  or  $1/q^2$ . Expanding the integrand in this variable gives rise to simpler integrals. These might suffer from spurious extra singularities (typically of IR type) which could question the convergence of the generated series. But the theory of asymptotic expansions of Feynman diagrams, now a textbook matter [11], allows to control this problem. Results of this approach can be found in [2, 3].

The other approach involves just a simple Taylor expansion below all thresholds and a subsequent analytic continuation of the resulting series. One uses a conformal mapping to enlarge the domain in which the expansion is valid. For the 2-point case one can map the branch cut onto a circle and the whole complex  $q^2$  plane into the interior of this circle, so that the results are valid for the whole  $q^2$  domain [4, 5].

One might add Padé approximants to improve the convergence of such series. In certain cases this method has been demonstrated to give excellent results [4, 5, 6].

At present these two approaches have been developed only for problems depending on one kinematical variable (e.g. 3-point function with one mass scale and two massless external particles). In fact, in both methods one can expect difficulties in the more general case.

The asymptotic expansions will suffer from an enormous proliferation of terms which correct the spurious infrared behaviour, while the analytic continuation of functions depending on several complex variables could possibly constitute a serious mathematical problem for the second method [12].

Another approach consists in the expansion of some of the involved propagators according to  $x$ -space or Mellin-Barnes type representations [7, 8]. Again, one will end up with series representations valid in certain kinematical domains.

One case which can be solved using any of the above methods is the two-point master function of Fig. 1. While these methods were being developed, almost all made use of yet another representation, obtained by one of us [13], to check their results. In [13], a very simple integral representation was obtained, which allowed a straightforward numerical integration. One can even use this approach to obtain results for general tensor integrals [14, 15] or to obtain integral representations for graphs directly [15].

Following the ideas of [13, 16] we present in this paper an approach which is modest in its use of computer power, is of comprehensive structure, and is valid for arbitrary kinematical regions. The principle of our method is the same for 2- and 3-point functions, and in the following section we explain it in much detail with the example of the propagator function. In this way we avoid unnecessary complications stemming from a more complex analytical structure of the vertex. We pay special attention to separate calculation of real and imaginary parts. The extension to the 3-point functions is presented in section 3. In section 4 we provide two examples which illustrate applications of our method. Both examples refer to two-loop planar vertex functions which have been known from other approaches in some special cases, and therefore allow a comparison of accuracy and flexibility of various methods. The last section summarizes our results.

## 2 Propagator function

An exposition of our method is simpler with the example of the propagator function, so we are going to present it in this section in much detail. The two-loop propagator function is depicted in Fig. 1 and we normalize it as in [13, 17]:

$$I(q^2) = -\frac{q^2}{\pi^4} \int d^4l \int d^4k \frac{1}{P_1 P_2 P_3 P_4 P_5}, \quad (1)$$

where  $q$  is the four-momentum of the external particle,  $l$  and  $k$  are the internal loop momenta, and  $P_i$  are the propagators of the internal particles, as labeled in Fig. 1. We assume that the external particle is massive (the massless limit is easy to perform), and choose its rest frame for the calculation. The choice of parallel and orthogonal subspaces is Lorentz invariant, and therefore the calculation could be done in any Lorentz frame. The four-momenta can be chosen explicitly as

$$\begin{aligned} q^\mu &= (q, 0, 0, 0), \\ l^\mu &= (l_0 + l_1, l_1, \vec{l}_\perp), \\ k^\mu &= (k_0 + k_1, k_1, \vec{k}_\perp). \end{aligned} \quad (2)$$

The splitting of the integration space  $(l, k)$  into the subspaces orthogonal and parallel to the four-momentum of the external particle was the underlying idea of the method of doing two-loop integrals proposed by one of us [13, 18]. Our present method differs from that one in that we parametrize the orthogonal spaces using cylindrical rather than spherical coordinates, which leads to very different structure of the integrals. Such parametrization is necessary in the case of the vertex function, as will be seen in the following section, because there the axial coordinate of the cylinder ( $k_1$  or  $l_1$ ) belongs to the parallel space [16]. The advantage of using cylindrical coordinates for the propagator function is that it enables us to separate the real and imaginary parts of the function analytically and compute them separately, so that we avoid using complex arithmetics in the numerical program. The structure of kinematical singularities also becomes more transparent in this parametrization.

The propagators  $P_i$  of the internal particles can be written down using the explicit form of the four-momenta given in eq. (2):

$$\begin{aligned} P_1 &= l_0^2 + 2l_1 l_0 - l_\perp^2 - m_1^2 + i\eta, \\ P_2 &= (l_0 - q)^2 + 2l_1(l_0 - q) - l_\perp^2 - m_2^2 + i\eta, \\ P_3 &= (l_0 + k_0)^2 + 2(l_1 + k_1)(l_0 + k_0) - l_\perp^2 - k_\perp^2 - m_3^2 - 2l_\perp k_\perp z + i\eta, \\ P_4 &= k_0^2 + 2k_1 k_0 - k_\perp^2 - m_4^2 + i\eta, \\ P_5 &= (k_0 + q)^2 + 2k_1(k_0 + q) - k_\perp^2 - m_5^2 + i\eta, \end{aligned} \quad (3)$$

where  $z$  is the cosine of the angle between  $\vec{l}_\perp$  and  $\vec{k}_\perp$ , and  $-i\eta$  is a small imaginary part assigned to the masses of internal particles.

The volume element in the integral in eq. (1) can be rewritten as

$$d^4k d^4l = \frac{1}{4} dl_0 dk_0 dl_1 dk_1 ds dt d\alpha \frac{dz}{\sqrt{1 - z^2}}, \quad (4)$$

with  $s \equiv l_\perp^2$  and  $t \equiv k_\perp^2$ , and the angle  $\alpha$  describing the absolute position of  $\vec{l}_\perp$  and  $\vec{k}_\perp$ . The integration over  $\alpha$  is trivial and gives an overall factor  $2\pi$ . Of all internal propagators only  $P_3$  depends on  $z$ , and can be written as  $P_3 = A + Bz$ , so that after the integration over  $z$  using

$$\int_{-1}^1 \frac{dz}{\sqrt{1-z^2}} \frac{1}{A+Bz} = \frac{\pi}{\sqrt{A^2-B^2}}, \quad (5)$$

the propagator function becomes equal:

$$\begin{aligned} I(q^2) &= -\frac{q^2}{2\pi^2} \int \frac{dl_0 dk_0 dl_1 dk_1 ds dt}{\sqrt{A^2-B^2}} \frac{1}{P_1 P_2 P_4 P_5} \\ &= -\frac{q^2}{2\pi^2} \int \frac{dl_0 dk_0 dl_1 dk_1 ds dt}{\sqrt{A^2-B^2}} \\ &\quad \frac{1}{P_1 - P_2} \frac{1}{P_4 - P_5} \left( \frac{1}{P_2 P_5} - \frac{1}{P_2 P_4} - \frac{1}{P_1 P_5} + \frac{1}{P_1 P_4} \right) \end{aligned} \quad (6)$$

In the next step we integrate over  $k_1$  and  $l_1$  using the Cauchy theorem. The Fubini theorem allows the change of the order of integration, as the integral over the modulus of the integrand exists. This is true even in the degenerate limit  $q \rightarrow 0$  where  $(P_1 - P_2)$  and  $(P_4 - P_5)$  are independent of any integration variables. In order to determine which terms contribute we consider the analytical structure of the integrand. As explained in ref. [16], the position of cuts of the square root  $\sqrt{A^2 - B^2}$  as a function of  $l_1$  and  $k_1$  is determined by the sign of  $l_0 + k_0$ ; they are either both in the upper half-planes or both in the lower ones, for  $l_0 + k_0 < 0$  and  $l_0 + k_0 > 0$  respectively. Let's first take  $l_0 + k_0 > 0$ , so that we have to close the contours of  $k_1$  and  $l_1$  integrations in the upper half-planes. The following propagators have poles in this half-plane:  $P_1$  if  $l_0 < 0$ ,  $P_2$  if  $l_0 - q < 0$ ,  $P_4$  if  $k_0 < 0$ , and  $P_5$  if  $k_0 + q < 0$ . Not all of these conditions can be reconciled with the inequality  $l_0 + k_0 > 0$ , and in fact only the term  $1/(P_2 P_4)$  contributes. Analogously, for  $l_0 + k_0 < 0$ , only  $1/(P_1 P_5)$  contributes. Moreover, in each case the three inequalities which must be satisfied restrict the region of integration to a triangle in the  $(k_0, l_0)$  plane. Integration over the rest of the plane gives identically zero. The relevant triangles for positive and negative  $l_0 + k_0$  are depicted in Fig. 2(a) and (b), respectively.

After the  $k_1$  and  $l_1$  integrations the propagator function is represented by:

$$\begin{aligned} I(q^2) &= -\frac{q^2}{2} \left( \iint_{T_a} \frac{1}{k_0(l_0 - q)} + \iint_{T_b} \frac{1}{l_0(k_0 + q)} \right) dl_0 dk_0 \int_0^\infty ds \int_0^\infty dt \\ &\quad \frac{1}{P_1 - P_2} \frac{1}{P_4 - P_5} \frac{1}{\sqrt{A^2 - B^2}} \\ &= \frac{1}{2} \left( \iint_{T_a} + \iint_{T_b} \right) dl_0 dk_0 \int_0^\infty \frac{ds}{s + s_0 - i\eta} \int_0^\infty \frac{dt}{t + t_0 - i\eta} \\ &\quad \frac{1}{\sqrt{(at + b + cs)^2 - 4st}}, \end{aligned} \quad (7)$$

with  $T_a$  and  $T_b$  representing the triangles depicted in Fig. 2, and  $a$ ,  $b$  and  $c$  being functions of masses  $m_i$  and momenta  $k_0$  and  $l_0$ . These functions, although in general quite involved, can be easily calculated with any algebraic manipulation program by inserting expressions for the position of poles in  $l_1$  and  $k_1$  into the quantity  $A$  obtained from  $P_3 (= A + Bz)$ . Of course, these coefficients, as well as explicit forms of the propagators  $P_i$  have to be calculated separately for each of the triangles  $T_a$  and  $T_b$ . We give the explicit formulas in the Appendix; the only important fact we need to now for further derivations is that the coefficient functions  $a$  and  $c$  are always negative in the domain of integration. Finally, the structure of quantities  $s_0$  and  $t_0$  is essential for the calculation of the imaginary part of the self-energy diagram. Their explicit form will be given in the following section.

## 2.1 Imaginary part of the propagator function

In equation (7) one integrates only over positive values of  $s$  and  $t$ , so the contributions to the imaginary part occur either if  $s_0$  or  $t_0$  are negative, or if the argument of the square root becomes negative. These two possibilities correspond to cuts of the diagram in Fig. 1 across two and three internal lines, respectively. We first discuss the cut across two internal lines. In the general mass case, the explicit forms of  $s_0$  and  $t_0$  are:

$$\begin{aligned} s_0 &= l_0^2 + \frac{m_2^2 - m_1^2 - q^2}{q} l_0 + m_1^2, \\ t_0 &= k_0^2 - \frac{m_5^2 - m_4^2 - q^2}{q} k_0 + m_4^2. \end{aligned} \quad (8)$$

The condition that  $s_0$  ( $t_0$ ) be negative is a quadratic inequality for  $l_0$  ( $k_0$ ). The solutions describe stripes in the  $(k_0, l_0)$  plane in which the integral gets an imaginary part. These regions are depicted in Fig. 3. It can be seen that real solutions for  $l_0$  and  $k_0$  exist if  $q$  satisfies

$$q^2 > (m_i + m_j)^2, \quad (9)$$

$$\text{or} \quad q^2 < (m_i - m_j)^2, \quad (10)$$

where  $i = 1, j = 2$  for  $s_0$ , and  $i = 4, j = 5$  for  $t_0$ . The condition (9) describes a normal threshold corresponding to a cut across the lines 1 and 2 or 4 and 5 in the diagram in Fig. 1, whereas (10) corresponds to a pseudothreshold; it can be seen that it leads to solutions for  $k_0$  and  $l_0$  outside the integration triangles.

If  $s_0$  or  $t_0$  are negative we rewrite the integral representation in the form:

$$\begin{aligned} \text{Im } I(q^2) &= \frac{\pi}{2} \left( \iint_{T_a} + \iint_{T_b} \right) dl_0 dk_0 \left\{ \text{P.V.} \int_0^\infty \frac{ds}{s + s_0} \int_0^\infty dt \delta(t + t_0) \right. \\ &\quad \left. + \text{P.V.} \int_0^\infty \frac{dt}{t + t_0} \int_0^\infty ds \delta(s + s_0) \right\} \frac{1}{\sqrt{(at + b + cs)^2 - 4st}}. \end{aligned} \quad (11)$$

In each case one integration cancels a delta function and the other is elementary and can be done with Euler's change of variables [19].

In order to analyze the contribution to the imaginary part from the square root in eq. (7), we have to find the four-dimensional region in variables  $(s, t, l_0, k_0)$  where the function  $(at + b + cs)^2 - 4st$  is negative. The projection of such region on the  $(s, t)$  plane is described by an ellipse:

$$s = \frac{1}{c^2} \left[ \sqrt{t} \pm \sqrt{t(1 - ac) - bc} \right]^2, \quad (12)$$

from which it follows that the condition for the imaginary part to occur in this case is  $b > 0$  (the expression  $1 - ac$  is always negative in the region of integration). By an explicit calculation we convince ourselves that the inequality  $b > 0$  describes a closed region inside the triangular regions of integration (see Fig. 3), provided that at least one of the threshold conditions  $q^2 > (m_1 + m_3 + m_5)^2$  or  $q^2 > (m_2 + m_3 + m_4)^2$  are satisfied. As already indicated, they correspond to the three particle intermediate states, or cuts across the lines 1, 3, 5 or 2, 3, 4. Also in this case there exist pseudothreshold solutions beyond the limits of the integration region.

The integrations over  $s$  and  $t$  in this case can be done by observing that the square root in the integrand vanishes on the boundary of the integration region, and one can therefore use the formula analogous to eq. (5).

Finally, we note that if  $m_3=0$ , both types of imaginary contributions are divergent. Their sum, however, is free from this infrared divergence.

## 2.2 The real part

The calculation of the real part is conceptually simpler, since the general formula is just:

$$\begin{aligned} \text{Re } I(q^2) = & \frac{1}{2} \left( \iint_{T_a} + \iint_{T_b} \right) dl_0 dk_0 \text{P.V.} \int_0^\infty \frac{ds}{s + s_0} \text{P.V.} \int_0^\infty \frac{dt}{t + t_0} \frac{1}{\sqrt{(at + b + cs)^2 - 4st}} \\ & - \frac{\pi^2}{2} \iint_V dl_0 dk_0 \frac{1}{\sqrt{(at_0 - b + cs_0)^2 - 4s_0 t_0}}, \end{aligned} \quad (13)$$

where the second term originates from the region  $V$  of the  $(l_0, k_0)$  plane where we get a product of two imaginary quantities, yielding a contribution to the real part. This term is only present if  $q^2$  is above at least two thresholds, and the boundary of  $V$  can be determined from the condition that two imaginary contributions occur there. An example of such region can be seen in Fig. 3 as a triangle formed by a cross-section of two stripes corresponding to the two-particle cuts.

In the first term we have to perform a double integration over  $s$  and  $t$ . Two consecutive Euler's changes of variables reduce the problem to simple integrals of the form

$$\int \frac{\ln x}{x^2 + \alpha x + \beta} dx, \quad (14)$$

which can be expressed in terms of dilogarithms [20].

### 3 Vertex function

In the standard model calculations one encounters two topologies of two-loop vertex functions, as depicted in Fig. 4. In the present paper we consider only the planar case. The crossed diagram can be calculated using the same technique [16]; details and numerical results will be published separately.

Our approach to the calculation of the vertex function follows closely the method presented in the previous section in the context of the propagator. We present some details with the example of a decay of a particle of mass  $\sqrt{q^2}$  into particles with four-momenta  $q_1$  and  $q_2$ . For fixed values of  $q_1^2$  and  $q_2^2$  we define the vertex function as:

$$V(q^2) = \int \frac{d^4k d^4l}{P_1 P_2 P_3 P_4 P_5 P_6}. \quad (15)$$

With external momenta parametrized as

$$\begin{aligned} q_1 &= (q_{10}, q_z, 0, 0), \\ q_2 &= (q_{20}, -q_z, 0, 0), \end{aligned} \quad (16)$$

and the internal ones – as in the eq. (2), the propagators  $P_i$  have the following form:

$$\begin{aligned} P_1 &= l_0^2 + q_{10}^2 - q_z^2 + 2l_1(l_0 + q_{10} - q_z) + 2l_0q_{10} - l_\perp^2 - m_1^2 + i\eta, \\ P_2 &= l_0^2 + q_{20}^2 - q_z^2 + 2l_1(l_0 - q_{20} - q_z) - 2l_0q_{20} - l_\perp^2 - m_2^2 + i\eta, \\ P_3 &= (l_0 + k_0)^2 + 2(l_1 + k_1)(l_0 + k_0) - l_\perp^2 - k_\perp^2 - m_3^2 - 2l_\perp k_\perp z + i\eta, \\ P_4 &= k_0^2 + q_{10}^2 - q_z^2 + 2k_1(k_0 - q_{10} + q_z) - 2k_0q_{10} - k_\perp^2 - m_4^2 + i\eta, \\ P_5 &= k_0^2 + q_{20}^2 - q_z^2 + 2k_1(k_0 + q_{20} + q_z) + 2k_0q_{20} - k_\perp^2 - m_5^2 + i\eta, \\ P_6 &= k_0^2 + 2k_1k_0 - k_\perp^2 - m_6^2 + i\eta. \end{aligned} \quad (17)$$

Exactly as in the calculation of the propagator function, the angular integrations are done first, yielding the same type of a square root function, whose cuts determine how to close contours of  $l_1$  and  $k_1$  integrations. The only difference is that there are 6 propagators now and the partial fraction becomes slightly more complicated. Consequently, there are more terms which give nonzero contributions to the integral, and one has to distinguish four different integration domains in the  $(k_0, l_0)$  plane. They are depicted in Fig. 5, and we will label them accordingly  $T_{a,b,c,d}$ . The representation of the vertex function before splitting it into the imaginary and real parts can be written down in analogy to eq. 7:

$$\begin{aligned} V(q^2) &= \left( \iint_{T_a} + \iint_{T_b} + \iint_{T_c} + \iint_{T_d} \right) dl_0 dk_0 C(k_0, l_0) \\ &\quad \int_0^\infty \frac{ds}{s + s_0 - i\eta} \int_0^\infty \frac{dt}{(t + t_0 - i\eta)(t + t'_0 - i\eta)} \frac{1}{\sqrt{(at + b + cs)^2 - 4st}}, \end{aligned} \quad (18)$$

where  $C(k_0, l_0)$ ,  $s_0$ ,  $t_0$ ,  $t'_0$ ,  $a$ ,  $b$ ,  $c$  are rational functions of  $l_0$  and  $k_0$ , which have to be computed in each of the four integration regions  $T_i$  separately. The only difference from the analogous representation of the two-point function consists in an extra



$t$ -dependent term in the denominator, which arises because there are now four propagators in one of the loops, instead of three in the previous case. This new term does not cause any difficulties since we can decompose the integrand into simple fractions. The calculation of real and imaginary parts can now be done in the same manner as described in the previous section.

The appearance of an extra  $t$ -dependent term corresponds to the new way of cutting the diagram, across the lines 4 and 6 or 5 and 6, and reflects the richer analytical structure of the vertex diagram. Also the fact that there are 4 domains of integration in the present case is connected with four possible 3-particle cuts of the planar vertex. The square root in the integrands reflects in every domain presence of a distinct 3-particle threshold. This result is quite general (e.g., there are 2 domains in case of a 2-point function and 6 in the 3-point crossed vertex).

Finally we note that  $s$  and  $t$  integrations can be done analytically just like in the case of the 2-point function, so that we obtain a double integral representation for the vertex function.

## 4 Examples

The method which we have presented in this paper can be applied to the calculation of any two-loop two- or three- point function. In the present section we give examples of two classes of planar vertex functions for which results have been known in some limiting cases, and therefore it is possible to verify the computation. Both examples will also show that our method significantly extends the class of computable diagrams or improves the accuracy of the result.

As the first example we take a diagram of Fig. 4(a) with  $q_1^2 = q_2^2 = 0$ ,  $m_1 = m_2 = m_4 = m_5 = m_6 \equiv M$ , and study the real and imaginary parts of the corresponding amplitude at various values of  $q^2$  and  $m_3$ . For the particular value of  $m_3 = 0$  this diagram has been studied in refs. [1, 4, 5], so this limiting case is well known. The physical motivation for calculating this diagram is, e.g., the decay of a Higgs particle into two photons via a heavy quark triangle. The massless particle on line 3 represents a gluon. With the help of our method we can calculate this diagram for nonvanishing  $m_3$ , corresponding, for example, to an exchange of another Higgs particle inside the quark loop. The main complication with respect to the massless case is that there are now two thresholds on the positive real axis of variable  $q^2$ , corresponding to cuts across two and three internal lines. In fact the case of vanishing  $m_3$  is the hardest one to compute numerically, because the contributions from both cuts are infrared divergent. The limit is, however, approached smoothly, and one obtains excellent agreement with previous calculations already for  $m_3/M = 10^{-3}$ , for which results are shown in Fig. 6 (the real part) and in Fig. 7 (the imaginary part) as a solid line. The circles denote the points for which the results have been published in [5], and the long- and short- dashed lines depict the cases of  $m_3$  equal to  $M$  and  $2M$ , respectively. For easy comparison with refs. [1, 4, 5]<sup>2</sup> we take  $M = 150$  GeV, and plot the function

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<sup>2</sup>Ref. [4, 5] omits the factor  $10^9$

$I(q^2)$  defined as:

$$I(q^2) = 10^9 \pi^4 V(q^2), \quad (19)$$

with  $V(q^2)$  defined in eq. 15. As a cross check we have used a program based on dispersion relations and got excellent agreement, although the calculation of the real part is much more time consuming in the dispersion approach.

In our second example we examine a scalar diagram corresponding to a decay of a particle of mass  $\sqrt{q^2}$  into two heavy quarks, with two massive  $Z$  bosons exchanged between the outgoing quarks. In the notation of Fig. 4(a) we take  $\sqrt{q_1^2} = \sqrt{q_2^2} = m_1 = m_2 = m_4 = m_5 \equiv M$  and  $m_3 = m_6 = m_Z$ , where  $m_Z$  is the mass of the  $Z$  boson which we take equal 91.17 GeV. In this process the outgoing particles are massive, and no results have been obtained in such cases in the approaches using asymptotic expansions. Therefore we compare our results with the numbers obtained using Monte Carlo approach. The real part of this diagram is depicted in Fig. 8 for two masses of the heavy quark:  $M = 150$  GeV (solid line) and 174 GeV (dashed line). The dots represent data points of ref. [1]<sup>3</sup> obtained for  $M = 150$  GeV. Similarly, the imaginary part is plotted in Fig. 9. Our results lie within error bars indicated by the authors of ref. [1], our numerical error is, however, at least two orders of magnitude smaller. We have compared our results with numbers given by unitarity relations and obtained agreement up to one part in  $10^6$ , and even further refinement is possible.

## 5 Conclusions

In this paper we have presented a method of calculating two-loop two- and three-point functions, based on a double integral representation convenient for numerical evaluation. Although this method is applicable to arbitrary mass configurations, we introduced it by presenting relatively simple examples. Already these examples show that our method allows treating more general cases and obtain better accuracy than has been possible using other methods.

There are three directions in which our results can be generalized. First, one should consider mass cases in which other types of thresholds are present, and also include the crossed topology for the three-point functions (Fig. 4). This demands a more detailed investigation of the domains contributing to the imaginary part, but does not pose a principal problem.

Second, one can generalize this method easily to the case of four-point functions. Just as in the transition from two-point to vertex functions, the transition from vertex functions to box functions results in a proliferation of terms, while the general idea remains unchanged. Certainly, the three-dimensional parallel space requires an appropriate treatment, but the general structure of the integrands will not be altered. We will present these results elsewhere.

However, even when all scalar integrals are available, this is not the end of a practical two-loop calculation, and this brings us to the third line of generalizations. One still has to treat the tensor structure of a given graph; its subdivergent behaviour

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<sup>3</sup>We are very grateful to Dr. Fujimoto for sending us his numerical results.

has to be taken into account and one-loop counterterm diagrams have to be calculated; and one has to impose renormalization conditions which typically involve on-shell singularities.

In recent works [15, 21] one of us proposed a way of achieving all these aims using a concise algebraic algorithm. This algorithm is now being tested with simple examples, and we intend to demonstrate its use in standard model calculations in the future. Together with the results on UV-finite scalar two-loop functions obtained here, this paves the way to a convenient approach to two-loop Feynman graph calculations, especially for the standard model, where one is usually plagued by enormous tensor-algebraic and analytical difficulties, resulting from the interaction of fields with various masses and Lorentz properties.

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## Appendix

We list here explicit formulae for the coefficient functions  $a$ ,  $b$  and  $c$ , as defined in eq. (7), for the case of the 2-point function. There are two cases which we have to treat separately, depending on the residues of the propagators from which  $k_1$  and  $l_1$  were determined:

Case  $P_1P_5$

$$\begin{aligned} a &= \frac{l_0 - q}{k_0 + q}, \\ b &= \frac{1}{l_0(k_0 + q)} \left[ (m_1^2 - ql_0)(k_0^2 + k_0l_0 + k_0q + l_0q) \right. \\ &\quad \left. - m_3^2l_0(k_0 + q) + m_5^2l_0(k_0 + l_0) \right] + i\eta, \\ c &= \frac{k_0}{l_0}. \end{aligned} \tag{20}$$

Case  $P_2P_4$

$$a = \frac{l_0}{k_0},$$

$$\begin{aligned}
b &= \frac{1}{k_0(l_0 - q)} \left[ (m_4^2 + k_0 q)(l_0^2 + k_0 l_0 - k_0 q - l_0 q) \right. \\
&\quad \left. + m_2^2 k_0(k_0 + l_0) + m_3^2 k_0(q - l_0) \right] + i\eta, \\
c &= \frac{k_0 + q}{l_0 - q}.
\end{aligned} \tag{21}$$

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## Figure captions

Fig. 1: Two-loop propagator “master” diagram

Fig. 2: Regions of integration in the  $(k_0, l_0)$  plane

Fig. 3: Example of the structure of the integration region in the  $(k_0, l_0)$  plane: term arising from the residues in  $P_1$  and  $P_5$ , for  $m_1 = m_2 = m_4 = m_5$ . The shaded areas correspond to contributions of 2-particle cuts to the imaginary part. The irregular closed region depicts the area where the 3-particle state contributes.

Fig. 4: Two topologies of two-loop vertex diagrams

Fig. 5: Four integration domains for the vertex function

Fig. 6: Real part of the two-loop planar vertex function for  $m_i = M$  ( $i = 1, 2, 4, 5, 6$ ),  $m_3$  vanishing (solid line),  $m_3 = M$  (long dash), and  $m_3 = 2M$  (short dash).

Fig. 7: Imaginary part of the two-loop planar vertex function for  $m_i = M$  ( $i = 1, 2, 4, 5, 6$ ),  $m_3$  vanishing (solid line),  $m_3 = M$  (long dash), and  $m_3 = 2M$  (short dash).

Fig. 8: Real part of the two-loop planar vertex function for  $\sqrt{q_1^2} = \sqrt{q_2^2} = m_1 = m_2 = m_4 = m_5 = M$  and  $m_3 = m_6 = m_Z$ , for  $M = 150$  GeV (solid line), and  $M = 174$  GeV (dashed line).

Fig. 9: Imaginary part of the two-loop planar vertex function for  $\sqrt{q_1^2} = \sqrt{q_2^2} = m_1 = m_2 = m_4 = m_5 = M$  and  $m_3 = m_6 = m_Z$ , for  $M = 150$  GeV (solid line), and  $M = 174$  GeV (dashed line).